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FLOW AND HEAT TRANSFER OF AN ANOMALOUSLY VISCOUS FLUID IN THE GAP BETWEEN ROTATION AND STATIONARY DISKS WITH NONUNIFORM PRESSURE ABOUT THE PERIMETER

V. M. Shapovalov, N. V. Tyabin, and L. M. Beder

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The problem of bypass flow of a liquid under the influence of nonuniform pressure about the perimeter is solved.

One of the more promising pumps for transporting melts of polymers and high-viscosity liquids is the so-called circular pump [1]. It has several technicoeconomic advantages over conventionl screw pumps, particularly high efficiency. The pressure about the perimeter of a circular pump is nonuniform, which results in bypass flow of the liquid in the space between the body and the end of the rotor. It is interesting to evaluate the size of this flow, since it affects the overall efficiency of the pump. Also, analysis of this type of flow may prove useful in the design of precision-metering spur-gear pumps, for which stability of flow rate is very important. The end seal can be regarded as a disk–disk system in which the liquid is subjected to intensive shear strains. A flow diagram is presented in Fig. 1. The top disk (pump body) is stationary, while the bottom disk (pump rotor) rotates with an angular velocity ω. We will ignore the hydrodynamic effect of the shaft. A bridge separating the intake and delivery zones is located at the point $r = R$, $\varphi = 0$. The bridge has negligibly small angular dimensions and its hydrodynamic effect can be ignored. The velocity field is a three-dimensional shear field.

Considering the condition $h \ll R$, we assume that creeping flow is realized in the gap, and the forces of gravity and inertia can be ignored. Here, $\partial P/\partial z \approx 0$, and there is no flow in the z direction. With allowance for these assumptions, the following boundary-value problem is formulated:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial z}\left(\mu \frac{\partial v_r}{\partial z}\right), \tag{1}$$

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$$\frac{1}{r}\frac{\partial P}{\partial \varphi} = \frac{\partial}{\partial z}\left(\mu\frac{\partial v_\varphi}{\partial z}\right), \qquad (2)$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} = 0, \qquad (3)$$

$$z = h,\ v_\varphi = v_r = 0,\ z = 0,\ v_\varphi = \omega r,\ v_r = 0,\ r = R,\ P = P_1(\varphi).$$

We assume that the rheological properties of the medium are characterized by the equation

$$\mu = \mu_0\left(\frac{I_2}{2}\right)^{\frac{n-1}{2}}.$$

Considering that the rotational velocity of the liquid is significantly greater than its radial velocity:

$$v_\varphi \gg v_r, \qquad (4)$$

we have the following for viscosity

$$\mu = \mu_0\left|\frac{\partial v_\varphi}{\partial z}\right|^{n-1}. \qquad (5)$$

We determine the component v_φ as the sum of functions

$$v_\varphi = V(z,\ r,\ \varphi) + W(z,\ r), \qquad (6)$$

where V is due (as will be shown below) to the nonuniformity of the pressure about the perimeter and W is due to the rotation of the bottom disk.

Having inserted (5) and (6) into (2), we obtain

$$\frac{1}{\mu_0 r}\frac{\partial P}{\partial \varphi} = \frac{\partial}{\partial z}\left[\left|\frac{\partial V}{\partial z} + \frac{\partial W}{\partial z}\right|^{n-1}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial z}\right)\right]. \qquad (7)$$

We assume that the inequality W >> V is valid for the velocity components (6). Considering this, we can write Eq. (7) in the form

$$\frac{1}{\mu_0 r}\frac{\partial P}{\partial \varphi} = n\frac{\partial}{\partial z}\left(\left|\frac{\partial W}{\partial z}\right|^{n-1}\frac{\partial V}{\partial z}\right) + \frac{\partial}{\partial z}\left(\left|\frac{\partial W}{\partial z}\right|^{n-1}\frac{\partial W}{\partial z}\right). \qquad (8)$$

In accordance with (8), the effects of the viscosity anomaly are due to the dominant component W. In obtaining (8), we took the first two terms of the expansion of function (5). The left side and the first term of the right side of Eq. (8) are independent of φ, so we reason that the component W is determined by the equation

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial W}{\partial z}\right)^n\right] = 0,$$

for the satisfaction of which it is sufficient to set $\partial^2 W/\partial z^2 = 0$. Here, to determine V we have the equation

$$\frac{1}{\mu_0 r}\frac{\partial P}{\partial \varphi} = n\frac{\partial}{\partial z}\left(\left|\frac{\partial W}{\partial z}\right|^{n-1}\frac{\partial V}{\partial z}\right).$$

With allowance for the equations obtained, problem (1)-(3) can be broken down into two problems: the first boundary-value problem

$$\partial^2 W/\partial z_2 = 0,\ z = h \quad W = 0, \quad z = 0 \quad W = \omega r; \qquad (9)$$

while the second

$$\frac{1}{\mu_0}\frac{\partial P}{\partial r} = \frac{\partial}{\partial z}\left(\left|\frac{\partial W}{\partial z}\right|^{n-1}\frac{\partial v_r}{\partial z}\right), \qquad (10)$$

$$\frac{1}{\mu_0 r}\frac{\partial P}{\partial \varphi} = n\frac{\partial}{\partial z}\left(\left|\frac{\partial W}{\partial z}\right|^{n-1}\frac{\partial V}{\partial z}\right), \qquad (11)$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial V}{\partial \varphi} = 0, \qquad (12)$$

$$z = h, \quad V = v_r = 0, \qquad (13)$$

$$z=0,\quad V=v_r=0, \tag{14}$$

$$r=R,\quad P=P_1(\varphi). \tag{15}$$

Equations (9) characterize flow of the fluid in a disk–disk viscometer and, according to [2], for W we can write

$$W=\omega r\left(1-\frac{z}{h}\right). \tag{16}$$

The right sides of Eqs. (10) and (11) are independent of z. Thus, having inserted (16) into (10) and (11), and having integrated over z, with allowance for (13) and (14) we obtain

$$v_r=-\frac{z(h-z)}{2\mu_0}\left(\frac{h}{\omega r}\right)^{n-1}\frac{\partial P}{\partial r}, \tag{17}$$

$$V=-\frac{z(h-z)}{2\mu_0 rn}\left(\frac{h}{\omega r}\right)^{n-1}\frac{\partial P}{\partial \varphi}. \tag{18}$$

The function $P(r, \varphi)$ in (17) and (18) is unknown. Having inserted (17) and (18) into Eq. (12), we obtain an equation for P:

$$\frac{\partial^2 P}{\partial r^2}+(2-n)\frac{1}{r}\frac{\partial P}{\partial r}+\frac{1}{nr^2}\frac{\partial^2 P}{\partial \varphi^2}=0, \tag{19}$$

which at $n = 1$ becomes the Laplace equation. In the general case ($n \neq 1$), creeping flow in the components V and v_r is not the Hill–Shaw type of flow [3]. The pressure distribution in the gap depends on the flow index n. The solution of Eq. (19), with allowance for condition (15) and the boundedness condition $P < \infty$ at $r = 0$, has the following form when obtained by the Fourier method [4]:

$$\frac{P}{P_m}=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^{\frac{n-1}{2}+\sqrt{\frac{(1-n)^2}{4}+\frac{k^2}{n}}}(a_k\cos k\varphi+\beta_k\sin k\varphi), \tag{20}$$

where a_k and β_k are Fourier coefficients of the function P_1/P_m:

$$a_k=\frac{1}{\pi}\int_0^{2\pi}\frac{P_1}{P_m}\cos k\varphi d\varphi,\quad \beta_k=\frac{1}{\pi}\int_0^{2\pi}\frac{P_1}{P_m}\sin k\varphi d\varphi \quad \text{for}\quad k\geqslant 1.$$

The ratio P/P_m in (20) characterizes the dimensionless pressure ($P/P_m \leqq 1$ if we take as P_m the difference between the maximum and minimum pressures on the perimeter and if we regard P and P_1 as the excess pressures relative to the minimum pressure). We assume that the function P_1 satisfies the periodicity condition

$$P_1(\varphi+2\pi)=P_1(\varphi).$$

Inserting (20) into (17) and (18), we obtain

$$v_r=-\frac{z(h-z)P_m}{2R\mu_0}\left(\frac{h}{\omega r}\right)^{n-1}\sum_{k=1}^{\infty}\left(\frac{n-1}{2}+\sqrt{\frac{(1-n)^2}{4}+\frac{k^2}{n}}\right)(a_k\cos k\varphi+\beta_k\sin k\varphi\left(\frac{r}{R}\right)^{\frac{n-3}{2}+\sqrt{\frac{(1-n)^2}{4}+\frac{k^2}{n}}},$$

$$V=-\frac{z(h-z)P_m}{2\mu_0 rn}\left(\frac{h}{\omega r}\right)^{n-1}\sum_{k=1}^{\infty}k\left(\frac{r}{R}\right)^{\frac{n-1}{2}+\sqrt{\frac{(1-n)^2}{4}+\frac{k^2}{n}}}(\beta_k\cos k\varphi-a_k\sin k\varphi). \tag{22}$$

According to (21), the velocity profile in the cross section of the slit is parabolic, while the profile of v_φ is made up of the linear profile (16) and parabolic profile (22).

The overflow from the delivery zone to the intake zone can be determined by using the component v_r. A circular pump is characterized by a monotonically increasing pressure distribution about the perimeter, as shown in Fig. 1. Here, the diagram of radial velocity on the per-

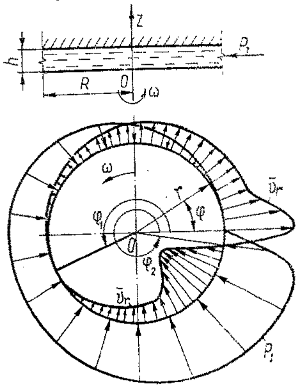

Fig. 1

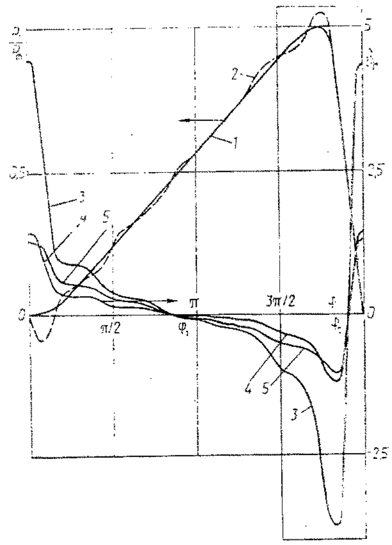

Fig. 2

Fig. 1. Diagram of fluid flow in the seal of a circular pump.

Fig. 2. Distribution of dimensionless parameters over the perimeter of the disk gap: 1) a priori assigned pressure distribution; 2) approximation by a trigonometric polynomial; 3-5) diagrams of dimensionless velocity $\hat{v}_r$ with different values of the flow index: n = 0.1, 1.0, 2.0.

imeter changes sign. The liquid flows out of the gap in the sector $\varphi_2 - 2\pi < \varphi < \varphi_1$ and flows into it in the sector $\varphi_1 < \varphi < \varphi_2$. The volumetric flow rates of the inflowing and outflowing liquids are equal and determine the rate of bypass flow Q.

The radial velocity on the perimeter, averaged over the height of the gap, is determined as the integral

$$\bar{v}_r = \frac{1}{h}\int_0^h v_r(r = R)\,dz$$

or, with allowance for (21),

$$\bar{v}_r = -\frac{h^2 P_m}{12R\mu_0}\left(\frac{h}{\omega R}\right)^{n-1}\sum_{k=1}^{\infty}\left(\frac{n-1}{2} + \sqrt{\frac{(1-n)^2}{4} + \frac{k^2}{n}}\right)(a_k \cos k\varphi + \beta_k \sin k\varphi). \qquad (23)$$

The condition for calculating the angles φ_1 and φ_2 has the form

$$\varphi = \varphi_{1,2}, \quad \bar{v}_r = 0.$$

The volumetric rate of bypass flow in the sector $\varphi_1 \leqslant \varphi \leqslant \varphi_2$ is characterized by the integral

$$Q = Rh\int_{\varphi_1}^{\varphi_2} \bar{v}_r d\varphi$$

or, with allowance for (23),

$$Q = -\frac{h^3 P_m}{12\mu_0}\left(\frac{h}{\omega R}\right)^{n-1}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{n-1}{2} + \sqrt{\frac{(1-n)^2}{4} + \frac{k^2}{n}}\right)[a_k(\sin k\varphi_2 - \sin k\varphi_1) - \beta_k(\cos k\varphi_2 - \cos k\varphi_1)]. \quad (24)$$

Let us find the frictional moment of the rotor. Let the shear stress in the gap be determined by the expression

$$\tau_{z\varphi} = \mu_0 \left| \frac{\partial W}{\partial z} \right|^{n-1} \frac{\partial v_\varphi}{\partial z} . \tag{25}$$

Integrating (25) over the surface of the bottom disk (z = 0), we find

$$M = \mu_0 R^3 \int_0^{2\pi} \int_0^1 \left(\left| \frac{\partial W}{\partial z} \right|^{n-1} \frac{\partial v_\varphi}{\partial z} \right) \Big|_{z=0} \left(\frac{r}{R} \right)^2 d \left(\frac{r}{R} \right) d\varphi. \tag{26}$$

Having inserted (6), (16), and (18) into (26), we write

$$M = \mu_0 R^3 \left\{ - \left(\frac{\omega R}{h} \right)^n \int_0^{2\pi} \int_0^1 \left(\frac{r}{R} \right)^{n+2} d \left(\frac{r}{R} \right) d\varphi - \frac{h}{2\mu_0 R n} \int_0^{2\pi} \int_0^1 \frac{\partial P}{\partial \varphi} \left(\frac{r}{R} \right) d \left(\frac{r}{R} \right) d\varphi \right\} .$$

Having integrated the first expression in the right side and having changed the order of integration in the second, we obtain

$$M = - \frac{2\pi \mu_0 R^3}{n+3} \left(\frac{\omega R}{h} \right)^n - \frac{R^2 h}{2n} \int_0^1 [P(\varphi = 2\pi) - P(\varphi = 0)] \left(\frac{r}{R} \right) d \left(\frac{r}{R} \right) .$$

Due to the periodicity of the function P, the second term is equal to zero. Thus, circular overflow of the liquid (due to the pressure nonuniformity) does not create a resulting turning moment. Finally, we have the following for the turning moment and the intake power $N = |M\omega|$:

$$M = - \frac{2\pi \mu_0 R^3}{n+3} \left(\frac{\omega R}{h} \right)^n, \quad N = \frac{2\pi \mu_0 R^3 \omega}{n+3} \left(\frac{\omega R}{h} \right)^n .$$

Let the pressure distribution about the perimeter have the form shown in Fig. 2. Figure 2 also shows an approximation of an a priori relation with the use of the first six terms of Fourier series (20). The numerical values of the Fourier coefficients were found by the method in [5, p. 558].

Equation (23) is conveniently written in dimensionless form:

$$\hat{v}_r = - \sum_{k=1}^{\infty} \left(\frac{n-1}{2} + \sqrt{\frac{(1-n)^2}{4} + \frac{k^2}{n}} \right) (a_k \cos k\varphi + \beta_k \sin k\varphi),$$

where

$$\hat{v}_r = \frac{12 \bar{v}_r R \mu_0}{P_m h^2} \left(\frac{\omega R}{h} \right)^{n-1} .$$

Figure 2 shows the distribution of $\hat{v}_r$ about the perimeter for different values of the parameter n. It is apparent from the figure that the maximum inflow velocity is found between the pressure maximum and the bridge ($\varphi = 0$; 2π). The velocity $\hat{v}_r$ changes sign in the region of the perimeter $1.75\pi < \varphi < 2\pi$, so that circulatory flow of the liquid takes place in the vicinity of of the delivery pipe. The velocity $\hat{v}_r$ increases with a decrease in n and $\hat{v}_r \to \infty$ at $n \to 0$. The position of the points at which $\hat{v}_r = 0$ depends little on the flow index. Thus, with a change in n from 0.1 to 2, the value of φ_1 changes from 2.66 to 2.7384 rad and the value of φ_2 changes from 5.9952 to 5.9936 rad.

We write Eq. (24) in the form

$$Q = - \frac{h^3 P_m}{6\mu_0} \left(\frac{h}{\omega R} \right)^{n-1} \zeta(n),$$

where

$$\zeta = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{n-1}{2} + \sqrt{\frac{(1-n)^2}{4} + \frac{k^2}{n}} \right) \sin \frac{k(\varphi_2 - \varphi_1)}{2} \cos \frac{k(\varphi_1 + \varphi_2)}{2} \times \left[a_k + \beta_k \operatorname{tg} \frac{k(\varphi_1 + \varphi_2)}{2} \right] .$$

The value of ζ depends on the form of the function P_1 and the flow index. For the case being examined, ζ is approximated by the following expressions in the range $0.1 \leqq n \leqq 2$ with an error no greater than 5%: $\zeta = 0.46 + 0.13/n$ for $0.1 \leqq n \leqq 1$, while $\zeta = 0.19n^2 - 0.44n + 0.84$ for $1 \leqq n \leqq 2$. When n = 1.16, the function ζ has a minimum.

The validity of assumption (4) is confirmed by numerical estimates of the overflow rates for practical values of the parameters. On the average, the component v_φ is greater than v_r by two orders of magnitude. Since v_r and V are commensurate, then the inequality $W \gg V$ is also valid.

In the formulation used here, the effects of the viscosity anomaly are due to the shear field resulting from rotation of the rotor. Thus, theoretical relations (21), (22), (23), and (24) lose meaning at $\omega = 0$. If we replace the complex $\mu_0(\omega R/h)^{n-1}$ in these expressions by μ and regard μ as the effective viscosity, then they can be used for the case n = 1, including $\omega = 0$.

Intensive heating occurs in the seal gap. Estimates show that only a small amount of this heat is removed with the bypass flow. Most of the heat is removed through the disks. In the case of polymer melts of low thermal conductivity, overheating of the material may lead to its degradation. Let there be a stationary temperature field in the gap. Convective heat transfer will be ignored. With allowance for $R \gg h$ and (4), the Fourier–Kirchhoff equation has the form

$$\lambda \frac{\partial^2 T}{\partial z^2} = -\tau_{\varphi z} \frac{\partial v_\varphi}{\partial z}. \tag{27}$$

Assuming $v_\varphi \approx W$ and considering the boundary conditions $T(z = 0) = T(z = h) = T_d$, after integrating (27) twice over z, we obtain:

$$T = T_c + \frac{\mu_0}{2\lambda}\left(\frac{\omega r}{h}\right)^{n+1} z(h - z).$$

The greatest temperature increment $T - T_d$ occurs at the points with the coordinates $z = h/2$ and $r = R$:

$$T - T_c = \frac{\mu_0 h^{1-n}}{2\lambda}(\omega R)^{n+1}. \tag{28}$$

According to (28), at n = 1 the increase in temperature is independent of the gap height h.

NOTATION

R, ω, radius and angular velocity of rotor; r, z, φ, cylindrical coordinates; h, height of gap; P, pressure; μ, viscosity; μ_0, n, rheological parameters; I_2, second invariant of strain-rate tensor; v_r, v_φ, velocity components; V, W, components of velocity v_φ; P_1, pressure distribution on the perimeter of the disks; P_m, maximum pressure difference on the perimeter; k, number of harmonic; a_k, β_k, Fourier coefficients; φ_1, φ_2, angles of position of points on the perimeter at which $v_r = 0$; Q, rate of bypass flow; $\bar{v}_r$, radial velocity on the perimeter averaged over the gap height; $\tau_{z\varphi}$, shear stress; M, frictional moment; N, power; $\hat{v}_r$, dimensionless radial velocity on the perimeter; ξ, parameter dependent on the flow index; λ, thermal conductivity of the liquid; T, T_d, temperature of liquid and disks, respectively.

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